ABSTRACT

ANALYTIC PROPERTIES OF ZETA

AND L-FUNCTIONS

by

Steven Mark Gonek

Chairman: Hugh L. Montgomery

Our investigations focus on two aspects of the behavior of zeta and L-functions in the critical strip of the complex plane; that is, the region defined by $0 < \operatorname{Re} s < 1$.

In 1926 A. Ingham proved that as $T \to \infty$,

$$(1) \qquad \int_1^T |\zeta^{(\nu)}(\tfrac{1}{2} + it)|^2 dt \sim \frac{1}{2\nu+1} T(\log T)^{2\nu+1} ,$$

where $\zeta^{(o)}(s) = \zeta(s)$ (Riemann's zeta-function) and for $\nu = 1,2,\ldots,\zeta^{(\nu)}(s)$ is the ν^{th} derivative of $\zeta(s)$. In the first part of this dissertation we obtain new types of mean-value analogues of (1). For example, we prove unconditionally that if $\nu \geq 1$, then

$$\sum_{1 \leq \gamma \leq T} \zeta^{(\nu)}(\rho)\zeta^{(\nu)}(1-\rho) \sim A_\nu \frac{T}{2\pi}(\log T)^{2\nu+2}$$

as $T \to \infty$, where $\rho = \beta + i\gamma$ is the generic zero of $\zeta(s)$ and A_ν is a computable positive constant. If we assume

the Riemann hypothesis, this takes the form

$$(2)\qquad \sum_{1\le\gamma\le T} |\zeta^{(\nu)}(\tfrac{1}{2}+i\gamma)|^2 \sim A_\nu \frac{T}{2\pi}(\log T)^{2\nu+2}$$

as $T \to \infty$. It turns out that $A_1 = \frac{1}{12}$. Therefore, on dividing both sides of (1) by T and both sides of (2) by $\frac{T}{2\pi}\log T$ (the asymptotic number of zeros with $1 \le \gamma \le T$), we find that the mean-value of $|\zeta'(\frac{1}{2}+i\gamma)|^2$ for $\gamma \in [1,T]$ is one-fourth the mean-value of $|\zeta'(\frac{1}{2}+it)|^2$ in $[1,T]$. We also prove a discrete analogue of (1) corresponding to the case $\nu = 0$.

The point of departure for our second investigation is the remarkable extension by S.M. Voronin of H. Bohr's work on the value distribution of $\zeta(s)$ in the critical strip. Let D be a closed disc of radius $< \frac{1}{4}$ centered at $s = \frac{3}{4}$, and suppose $f(s)$ is continuous and non-vanishing on D and is analytic on the interior of D. Voronin recently showed that if $\varepsilon > 0$, there is a real number τ such that

$$\max_{s\varepsilon C} |\zeta(s+i\tau) - f(s)| < \varepsilon .$$

Thus, on the interior of D the translates of $\zeta(s)$ mimic every non-vanishing analytic function. By methods which are somewhat different from those used by Voronin, we prove results of this type for other zeta and L-functions and for more general sets than D.

Let C be a simply connected compact set in the strip

$\frac{1}{2} < \text{Re } s < 1$ and suppose that $f(s)$ is continuous and non-vanishing on C and analytic in the interior of C. If K is an abelian extension of the rationals and $\zeta_K(s)$ is the Dedekind zeta-function of K, then for any $\varepsilon > 0$, there exists a τ such that

$$\max_{s\varepsilon C} |\zeta_K(s+i\tau) - f(s)| < \varepsilon .$$

This result also holds with the Hurwitz zeta-function, $\zeta(s,\alpha)$, in place of $\zeta_K(s)$, where $0 < \alpha < 1, \alpha \neq \frac{1}{2}$, and α is either rational or transcendental. Moreover, the condition that $f(s)$ not vanish on C may be removed. Consequently, the real parts of the zeros of $\zeta(s,\alpha)$ are dense in $[\frac{1}{2},1]$. Previously, H. Davenport, H. Heilbronn, and J. Cassels showed that $\zeta(s,\alpha)$ has zeros in the half-plane $\text{Re } s > 1$ for $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$.

We prove two results for Dirichlet's L-functions, $L(s,\chi)$. One concerns simultaneous approximation by the set of all L-functions (mod q). The other is the following q-analogue. Let C, $f(s)$, and ε be as above. Then for all large q, there exists a character χ (mod q) such that

$$\max_{s\varepsilon C} |L(s,\chi) - f(s)| < \varepsilon .$$

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by
Steven Mark Gonek

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Doctoral Committee:

Professor Hugh L. Montgomery, Chairman
Professor Donald J. Lewis
Professor Roger Low
Professor George Piranian
Professor Joseph Ullman

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PREFACE

In this thesis our investigations focus on two different aspects of the behavior of zeta and L-functions in the critical strip of the complex plane; that is, the region defined by $0 < \text{Re } s < 1$.

In 1926 A. Ingham [8] proved that as $T \to \infty$,

$$(1) \qquad \int_1^T |\zeta^{(\nu)}(\tfrac{1}{2} + it)|^2 \, dt \sim \frac{1}{2\nu+1} T(\log T)^{2\nu+1} ,$$

where $\zeta^{(0)}(s) = \zeta(s)$ (Riemann's zeta-function) and for $\nu = 1,2,\ldots$, $\zeta^{(\nu)}(s)$ is the ν^{th} derivative of $\zeta(s)$. In Chapter I we obtain new types of mean-value theorems which are, when the Riemann hypothesis is assumed, discrete analogues of (1). For example, we prove unconditionally that if $\nu \geq 1$, then as $T \to \infty$,

$$\sum_{1 \leq \gamma \leq T} \zeta^{(\nu)}(\rho)\zeta^{(\nu)}(1-\rho) \sim A_\nu \frac{T}{2\pi}(\log T)^{2\nu+2}$$

where $\rho = \beta + i\gamma$ is the generic zero of $\zeta(s)$ and A_ν is a computable constant. On the Riemann hypothesis this takes the form

$$(2) \qquad \sum_{1 \leq \gamma \leq T} |\zeta^{(\nu)}(\tfrac{1}{2} + i\gamma)|^2 \sim A_\nu \frac{T}{2\pi}(\log T)^{2\nu+2}$$

A rather surprising consequence of (1) and (2) in the case

$\nu = 1$ is that the average size of $|\zeta'(\frac{1}{2}+i\gamma)|^2$ for $\gamma \in [1,T]$ is one-fourth the average size of $|\zeta'(\frac{1}{2}+it)|^2$ for arbitrary $t \in [1,T]$. We also prove a discrete analogue of (1) corresponding to the case $\nu = 0$.

The point of departure for our second investigation is a remarkable extension by S.M. Voronin of H. Bohr's work on the value distribution of $\zeta(s)$ in the critical strip. Let D be a closed disc of radius $< \frac{1}{4}$ centered at $s = \frac{3}{4}$, and suppose that $f(s)$ is continuous and non-vanishing on D and is analytic on the interior of D. Voronin [21] has shown that if $\varepsilon > 0$, there exists a real number τ such that

$$\max_{s\in D}|\zeta(s+i\tau) - f(s)| < \varepsilon .$$

This is a universality theorem for $\zeta(s)$ in that it asserts that the translates of $\zeta(s)$ approximate all the functions of some large class of functions. Our object in the second part of this thesis is to prove universality theorems for other zeta and L-functions and for more general sets than D. In Chapter II we establish a fundamental lemma upon which we base the proofs of all our universality theorems. In Chapter III we prove a "simultaneous" universality theorem for the Dirichlet L-functions to a given modulus. From this we deduce that the Dedekind zeta-function of any abelian extension of the rational number field is universal. In Chapter IV we show that certain of Hurwitz's zeta-functions are universal and that such functions posses zeros in the right half of the critical strip. Finally, in Chapter V, we prove a q-analogue of the universality of $\zeta(s)$.

Chapter I is completely independent of Chapters II-V, and conversely.

NOTATION

The notations "$\sim$", "O", "o", "$<<$", "$>>$" have their usual meanings.

The symbol [x] denotes the greatest integer $\leq x$, and x denotes the distance from x to the nearest integer, thus

$$x = \min_{n \in \mathbb{Z}} |x-n| .$$

We sometimes write $\exp(x)$ for e^x.

The letter $s = \sigma + it$ represents a complex variable. Unless otherwise stated, $\rho = \beta + i\gamma$ denotes a non-trivial zero of an L-function or of $\zeta(s)$.

We let p stand for a prime and q for a general modulus. The letter χ denotes a Dirichlet character except in Chapter I, where its alternate usage is explained. We write the arithmetic functions of Euler and von Mangoldt as usual; we write $d(n)$ for the divisor function. As is customary, $\pi(x)$ is the counting function of the primes and $\psi(x)$ is Chebychev's counting function,

$$\psi(x) = \sum_{n \leq x} \Lambda(n) .$$

The symbol $\sum_{\chi (\bmod q)}$ represents summation over all characters to the modulus q, and $\sum_{a=1}^{q}{}^{*}$ is shorthand for $\sum_{\substack{a=1 \\ (a,q)=1}}$.

In several places we have used similar notation for different things. For example, in Chapter I, $\psi(x)$ is Chebychev's function but $\psi(s)$ is the digamma function. In such instances the text makes clear which function is meant.

TABLE OF CONTENTS

CHAPTER I

MEAN-VALUES OF THE RIEMANN ZETA-FUNCTION AND ITS DERIVATIVES

§1. Statement of Results

In 1918 Hardy and Littlewood [7] proved that as $T \to \infty$

(1) $$\int_1^T |\zeta(\tfrac{1}{2} + it)|^2 \, dt \sim T \log T .$$

In 1926 Ingham [8] proved corresponding formulae for the derivatives of $\zeta(s)$. For example, he showed that as $T \to \infty$

(2) $$\int_1^T |\zeta^{(\nu)}(\tfrac{1}{2} + it)|^2 \, dt \sim \frac{T}{2\nu+1} (\log T)^{2\nu+1} ,$$

where ν is any non-negative integer, $\zeta^{(\nu)}(s)$ is the ν^{th} derivative of $\zeta(s)$, and $\zeta^{(0)}(s) = \zeta(s)$. Our object in this chapter is to prove some new types of mean-value theorems which are, when the Riemann hypothesis is assumed, discrete analogues of (1) and (2).

Our first result is

Theorem 1.1. Let $\rho = \beta + i\gamma$ be the generic non-trivial zero of $\zeta(s)$. If T is sufficiently large and c is any real number satisfying $|c| \le \frac{1}{4\pi} \log \frac{T}{2\pi}$, we have

$$(3) \quad \sum_{1\le\gamma\le T} \zeta(\rho + \frac{2\pi ic}{\log T/2\pi})\zeta(1 - \rho - \frac{2\pi ic}{\log T/2\pi})$$

$$= (1 - (\frac{\sin \pi c}{\pi c})^2) \frac{T}{2\pi}(\log T)^2 + O(T \log T) .$$

If the Riemann hypothesis is true, then

$$(4) \quad \sum_{1\le\gamma\le T} |\zeta(\frac{1}{2} + i\gamma + \frac{2\pi ic}{\log T/2\pi})|^2$$

$$= (1 - (\frac{\sin \pi c}{\pi c})^2) \frac{T}{2\pi}(\log T)^2 + O(T \log T) .$$

The error terms are independent of c.

For a fixed $c \neq 0$ (if $c = 0$ both (3) and (4) are trivial), (4) is a discrete analogue of (1). Since there are $\sim \frac{T}{2\pi} \log T$ zeros with $\gamma \in [1,T]$, it follows from (4) that the mean-square size of $\zeta(\frac{1}{2} + i\gamma + \frac{2\pi ic}{\log T/2\pi})$ over the zeros with $\gamma \in [1,T]$ is $\sim (1 - (\frac{\sin \pi c}{\pi c})^2) \log T$. On the other hand the mean-square size of $\zeta(\frac{1}{2} + it)$ over the entire interval $[1,T]$ is $\sim \log T$ by (1). The two means are equal if c is a non-zero integer. Otherwise the former mean is less than the latter.

We will also prove

Theorem 1.2. Let ρ be the generic non-trivial zero of $\zeta(s)$ and let μ,ν be positive integers. If T is sufficiently large, we have

(5) $$\sum_{1\le\gamma\le T} \zeta^{(\mu)}(\rho)\zeta^{(\nu)}(1-\rho)$$

$$= A(\mu,\nu)\frac{T}{2\pi}(\log T)^{\mu+\nu+2} + O(T(\log T)^{\mu+\nu+1}) ,$$

where

$$A(\mu,\nu) = (-1)^{\mu+\nu}\mu!\nu!\Big(\sum_{\kappa=0}^{\nu} \frac{(-1)^\kappa}{(\mu+\kappa+1)!(\nu-\kappa)!} \Big(\frac{1}{2} - \frac{1}{\mu+\kappa+2}\Big)$$

$$+ \sum_{\kappa=0}^{\mu} \frac{(-1)^\kappa}{(\nu+\kappa+1)!(\mu-\kappa)!} \Big(\frac{1}{2} - \frac{1}{\nu+\kappa+2}\Big) ,$$

and implicit constants may depend on μ and ν.

If the Riemann hypothesis is true and $\mu = \nu$, then

(6) $$\sum_{1\le\gamma\le T} |\zeta^{(\nu)}(\tfrac{1}{2} + i\gamma)|^2$$

$$= A(\nu,\nu)\frac{T}{2\pi}(\log T)^{2\nu+2} + O(T(\log T)^{2\nu+1}) .$$

Since the sum in (6) is over $\sim\frac{T}{2\pi}\log T$ zeros and $A(1,1) = \frac{1}{12}$, a comparison of (2) with (6) yields

Corollary 1.1. Assume the Riemann hypothesis. Then the mean-square size of $\zeta'(\frac{1}{2} + i\gamma)$ over the $\gamma \in [1,T]$ is one-fourth the mean-square size of $\zeta'(\frac{1}{2} + it)$ over all $t \in [1,T]$.

Of course, a simple way to account for the curious behavior of $\zeta'(s)$ described in the corollary is by assuming the existence of a positive density of multiple zeros of $\zeta(s)$.

However, one expects that all the zeros are simple.

We remark that we could replace the right-hand side of (6) by an expression of the form

$$A(\nu,\nu)\frac{T}{2\pi}P_{2\nu+2}(\log T) + O(T^{\theta}) ,$$

where $P_{2\nu+2}(\log T)$ is a monic polynomial of degree $2\nu+2$ in $\log\frac{T}{2\pi}$ and $\theta < 1$. Nevertheless we are content to prove (6) as it stands.

§2. Some Formulae and Estimates

Before we develop the basic ideas of the proofs of Theorems 1.1 and 1.2, it will be useful to set down certain formulae and estimates.

Let

$$\chi(1-s) = \pi^{1/2-s}\,\Gamma(\tfrac{s}{2})/\Gamma(\tfrac{1-s}{2}) , \tag{7}$$

$\Gamma(s)$ being the gamma-function. Using the well-known formulae (for example, see Whittaker and Watson [26; Chaps. 12, 13]), we see that

$$\Gamma(\tfrac{s}{2})/\Gamma(\tfrac{1-s}{2}) = \pi^{-1/2}\,2^{1-s}\cos\frac{s\pi}{2}\,\Gamma(s)$$

and

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} ;$$

we also find that

(8) $$\chi(1-s) = \frac{2^{1-s}\,\pi^{1-s}}{2\sin s\pi/2\;\Gamma(1-s)} .$$

We write Stirling's asymptotic formula for $\Gamma(s)$ in the simple form

(9) $$\log \Gamma(s) = (s-\tfrac{1}{2})\log s - s + \tfrac{1}{2}\log 2\pi + O(\tfrac{1}{|s|}) \qquad (|s| \geq \tfrac{1}{2}) .$$

This is valid for $-\pi+\delta < \arg s < \pi-\delta$ for any fixed $\delta > 0$ (see Whittaker and Watson [26; Chaps. 12, 13]). If $-\pi+\delta < \arg 1-s < \pi-\delta$, we may substitute $1-s$ for s in (9) and combine the result with (8) to obtain

(10) $$\chi(1-s) = (\tfrac{2\pi}{-s})^{1/2-s}\,\frac{e^{-s}}{2\sin s\pi/2}\,(1+O(\tfrac{1}{|1-s|})) \qquad (|1-s| \geq \tfrac{1}{2}) .$$

For fixed σ this takes the form

(11) $$\chi(1-s) = e^{-\pi i/4}\,(\tfrac{t}{2\pi})^{\sigma-1/2}\,\exp[it\log\tfrac{t}{2\pi e}]\,(1 + O(\tfrac{1}{|t|})) \qquad (t \geq 1) .$$

With $(1-s)$ as above, the unsymmetric form of the functional equation of $\zeta(s)$ is

(12) $$\zeta(1-s) = \chi(1-s)\,\zeta(s) .$$

We also require the symmetric form of the functional equation. Set

(13) $$\xi(s) = \tfrac{1}{2}\,s(s-1)\,\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,\zeta(s) .$$

Then we have

$$\xi(s) = \xi(1-s) \quad (14)$$

The function $\xi(s)$ is entire of order one and its only zeros are the non-trivial zeros of $\zeta(s)$.

Euler's psi-function is defined by

$$\psi(s) = \Gamma'(s)/\Gamma(s) \quad (15)$$

When $-\pi+\delta < \arg s < \pi-\delta$ and $|s| \geq \frac{1}{2}$, we have

$$\psi(s) = \log s + O\left(\frac{1}{|s|}\right) \quad (16)$$

This may be derived from (9) by means of Cauchy's inequality for analytic functions.

Consider the set of intervals defined by $\left\|\frac{t}{2\pi}\right\| \geq \frac{1}{100}$, $t \geq 0$; that is, the intervals of the form

$$I_n = [2\pi n + \frac{2\pi}{100}, 2\pi n + 99 \cdot \frac{2\pi}{100}] \qquad (n = 0, 1, \ldots) .$$

For $n \geq 2$, say, there are $\ll \log n$ ordinates of zeros of $\zeta(s)$ in I_n. Among these there must be a gap of length $\gg \frac{1}{\log n}$. Therefore we can choose a number T in I_n such that for each ordinate γ of a zero in I_n

$$|T - \gamma| \gg \frac{1}{\log T} .$$

Having selected one T in each I_n $(n=2,3,\ldots)$, we obtain a sequence tending to ∞ whose consecutive terms are $\ll 1$ apart. We denote this sequence by S . Recall that if T is large and does not coincide with the ordinate of any zero of $\zeta(s)$, then

$$\frac{\zeta'}{\zeta}(\sigma + iT) = \sum_{|\gamma - t|<1} \frac{1}{s-\rho} + O(\log T)$$

uniformly for $-1 \leq \sigma \leq 2$ (see Davenport [4; p.103]). If $T \in S$, each term in the sum is $\ll \log T$. As there are $\ll \log T$ terms we find

$$\frac{\zeta'}{\zeta}(\sigma + iT) \ll (\log T)^2 \tag{17}$$

for large $T \in S$ and uniformly for $-1 \leq \sigma \leq 2$. By logarithmic differentiation of (13) we have

$$\frac{\xi'}{\xi}(s) = \frac{\zeta'}{\zeta}(s) + \frac{1}{2}\psi\left(\frac{s}{2}\right) - \frac{1}{2}\log \pi + \frac{2s-1}{s(s-1)} \qquad (s \neq 1) . \tag{18}$$

We deduce from this, (16), and (17) that

$$\frac{\xi'}{\xi}(\sigma + iT) \ll (\log T)^2 \tag{19}$$

for large $T \in S$ and uniformly for $-1 \leq \sigma \leq 2$.

Similarly, we may combine the estimate

$$\frac{\zeta'}{\zeta}(\sigma + it) \ll \log 2|t|$$

valid for $\sigma \geq 1 - \frac{A}{\log 2|t|}$ and $|t| \geq 1$, where A is a positive absolute constant (see Titchmarsh [19; p.53]), with

(16) and (18) to obtain

$$\frac{\zeta'}{\zeta}(\sigma + it) \ll \log 2|t| \tag{20}$$

for $\sigma > 1 - \frac{A}{\log 2|t|}$, $|t| \geq 1$.

Finally, we need the estimates

$$\zeta^{(\nu)}(\sigma + it) \ll \begin{cases} |t|^{1/2-\sigma+\varepsilon} & \text{if } \sigma \leq 0 \\ |t|^{1/2(1-\sigma)+\varepsilon} & \text{if } 0 \leq \sigma \leq 1 \\ |t|^{\varepsilon} & \text{if } \sigma \geq 1 , \end{cases} \tag{21}$$

where $\varepsilon > 0$ is arbitrary, $|t| \geq \frac{1}{2}$, and $\nu \geq 0$ is an it-teger. These may be deduced from the classical estimates

$$\zeta(\sigma + it) \ll \begin{cases} |t|^{1/2-\sigma+\varepsilon} & \text{if } \sigma \leq 0 \\ |t|^{1/2(1-\sigma)+\varepsilon} & \text{if } 0 \leq \sigma \leq 1 \\ |t|^{\varepsilon} & \text{if } \sigma \geq 1 , \end{cases}$$

where $|t| \geq \frac{1}{4}$ (see Titchmarsh [19; pp.81-82]) by applying Cauchy's inequality for analytic functions to $\zeta(s)$ in a small disc centered at $s = \sigma + it$.

§3. Beginning of the Proofs

We can now begin the proofs of Theorems 1.1 and 1.2, although we shall require a section of lemmas (§4 below) in order to complete them.

Let a be a real number with $1 < a < 2$ and let D be the closed rectangle in the complex plane with vertices at $a+i$, $a+iT$, $1-a+iT$, $1-a+i$, where T is large. We define

$$(22)\quad I = I(\mu,\nu,i\delta) = \frac{1}{2\pi i}\int_{\partial D} \frac{\xi'}{\xi}(s)\zeta^{(\mu)}(s+i\delta)\zeta^{(\nu)}(1-s-i\delta)\,ds ,$$

where ∂D is the boundary of D and the integral is taken in the counterclockwise sense. We also assume that δ is real and $|\delta| \le \frac{1}{2}$. By the theory of residues it is clear that

$$(23)\qquad I(0,0,i\delta) = \sum_{1\le\gamma\le T} \zeta(\rho+i\delta)\zeta(1-\rho-i\delta)$$

and

$$(24)\qquad I(\mu,\nu,0) = \sum_{1\le\gamma\le T} \zeta^{(\mu)}(\rho)\zeta^{(\nu)}(1-\rho) ,$$

provided no zero ρ lies on ∂D . Since the ordinate of the first non-trivial zero of $\zeta(s)$ lying above the real axis is >14 and no zeros lie on the vertical edges of D , we need only insure that T is not the ordinate of any zero. This is the case if $T \in S$, the set constructed in §2. From now on we therefore assume $T \in S$. At the end of the proofs of Theorems 1.1 and 1.2 we will show that this entails no loss of generality.

To prove Theorems 1.1 and 1.2 we must estimate $I(\mu,\nu,0)$ and $I(0,0,i\delta)$. Some of the analysis can be done in common by working with the quantity $I(\mu,\nu,i\delta)$. It is this analysis which we describe now.

We split the integral $I(\mu,\nu,i\delta)$ into four integrals corresponding to the four sides of D . We write

$$I(\mu,\nu,i\delta) = \sum_{j=1}^{4} I_j(\mu,\nu,i\delta) ,$$

where I_1 is the integral over $[a+i, a+iT)$, I_2 is over $[a+iT, 1-a+iT)$, I_3 is over $[1-a+iT, 1-a+i)$, and I_4 is over $[1-a+i, a+i)$. Since we are assuming that $|\delta| \le \frac{1}{2}$, we have

$$I_4 \ll 1 .$$

For I_2 we find

$$\begin{aligned} I_2 &\ll \max_{1-a\le\sigma\le a} \left|\frac{\xi'}{\xi}(\sigma+iT)\zeta^{(\mu)}(\sigma+iT+i\delta)\zeta^{(\nu)}(1-\sigma-iT-i\delta)\right| \\ &\ll (\log T)^2 \max_{1-a\le\sigma\le a} |\zeta^{(\mu)}(\sigma+iT+i\delta)\zeta^{(\nu)}(1-\sigma-iT-i\delta) \end{aligned}$$

by virtue of (19). The last line is

$$\ll (\log T)^2 \Bigg(\max_{1-a\le\sigma<0} \left| \zeta^{(\mu)}(\sigma+iT+i\delta)\zeta^{(\nu)}(1-\sigma-iT-i\delta) \right|$$

$$+ \max_{0\le\sigma\le 1} \left| \zeta^{(\mu)}(\sigma+iT+i\delta)\zeta^{(\nu)}(1-\sigma-iT-i\delta) \right|$$

$$+ \max_{1<\sigma\le a} \left| \zeta^{(\mu)}(\sigma+iT+i\delta)\zeta^{(\nu)}(1-\sigma-iT-i\delta) \right| \Bigg) .$$

By (21) we see that

$$\max_{1-a\le\sigma<0} \left| \zeta^{(\mu)}(\sigma+iT+i\delta)\zeta^{(\nu)}(1-\sigma-iT-i\delta) \right|$$

$$\ll T^{\varepsilon} \max_{1-a\le\sigma<0} T^{\frac{1}{2}-\sigma+\varepsilon} = T^{a-\frac{1}{2}+2\varepsilon} .$$

Also

$$\max_{0\le\sigma\le 1} \left| \zeta^{(\mu)}(\sigma+iT+i\delta)\zeta^{(\nu)}(1-\sigma-iT-i\delta) \right|$$

$$\ll \max_{0\le\sigma\le 1} T^{\frac{1}{2}(1-\sigma)+\varepsilon}\, T^{\frac{1}{2}\sigma+\varepsilon} = T^{\frac{1}{2}+2\varepsilon} .$$

Similarly, we find that the maximum over $(1,a]$ is $\ll T^{a-\frac{1}{2}+2\varepsilon}$. Since $a-\frac{1}{2} > \frac{1}{2}$ and $\varepsilon > 0$ is arbitrarily small, we obtain

$$I_2 \ll T^{a-\frac{1}{2}+\varepsilon} .$$

This and the estimate for I_1 lead to

$$(25) \quad I(\mu,\nu,i\delta) = I_1(\mu,\nu,i\delta) + I_3(\mu,\nu,i\delta) + O(T^{a-\frac{1}{2}+\varepsilon}) .$$

We now treat

$$I_3(\mu,\nu,i\delta) = \frac{1}{2\pi i}\int_{1-a+iT}^{1-a+i} \frac{\xi'}{\xi}(s)\zeta^{(\mu)}(s+i\delta)\zeta^{(\nu)}(1-s-i\delta)ds \ .$$

The logarithmic derivative of (14) is

$$\frac{\xi'}{\xi}(s) = -\frac{\xi'}{\xi}(1-s) \ .$$

Using this and the fact that both $\zeta^{(\nu)}(s)$ and $\frac{\xi'}{\xi}(s)$ satisfy the reflection principle, we get

$$I_3(\mu,\nu,i\delta) = -\frac{1}{2\pi i}\int_{1-a+iT}^{1-a+i} \frac{\xi'}{\xi}(1-s)\zeta^{(\mu)}(s+i\delta)\zeta^{(\nu)}(1-s-i\delta)ds$$

$$= \frac{1}{2\pi i}\int_1^T \frac{\xi'}{\xi}(a-it)\zeta^{(\mu)}(1-a+it+i\delta)\zeta^{(\nu)}(a-it-i\delta)idt$$

$$= \overline{\frac{1}{2\pi i}\int_1^T \frac{\xi'}{\xi}(a+it)\zeta^{(\mu)}(1-a-it-i\delta)\zeta^{(\nu)}(a+it+i\delta)idt}$$

$$= \overline{\frac{1}{2\pi i}\int_{a+i}^{a+iT} \frac{\xi'}{\xi}(s)\zeta^{(\mu)}(1-s-i\delta)\zeta^{(\nu)}(s+i\delta)ds}$$

$$= \overline{I_1(\nu,\mu,i\delta)} \ .$$

This and (25) yield

$$(26) \quad I(\mu,\nu,i\delta) = I_1(\mu,\nu,i\delta) + \overline{I_1(\nu,\mu,i\delta)} + O(T^{a-\frac{1}{2}+\varepsilon}) \ .$$

Our problem is now reduced to estimating $I_1(\mu,\nu,i\delta)$ for arbitrary μ,ν when $\delta = 0$, and for $\mu = \nu = 0$ when $\delta \neq 0$. In the next section we prove the lemmas necessary for doing this.

4. Lemmas

Our first two lemmas are essentially due to N. Levinson [9; Lemmas 3.2-3].

Lemma 1.1. There is a small $c > 0$ such that

$$I_0 = \int_{r(1-c)}^{r(1+c)} \exp[it \log t/er](t/2\pi)^{a-1/2}\, dt$$

$$= (2\pi)^{1-a} r^a e^{-ir+\pi i/4} + O(r^{a-1/2})$$

for lárge r and a arbitrary but fixed.

Proof: Let $t = r(1+x)$ so that

$$I_0 = (2\pi)^{1/2-a} e^{ir} r^{a+1/2} I_1 ,$$

where

$$I_1 = \int_{-c}^{c} \exp[ir(1+x)\log(1+x) - irx](1+x)^{a-1/2}\, dx .$$

Let

$$I_2 = \int_{-c}^{c} \exp[ir(1+x)\log(1+x) - irx]\, dx .$$

Then

$$I_1 - I_2 =$$

$$\int_{-c}^{c} \exp[ir(1+x)\log(1+x)-irx]\log(1+x)\left\{\frac{\exp[(a-\frac{1}{2})\log(1+x)]-1}{\log(1+x)}\right\} dx.$$

An integration by parts shows that

$$I_1 - I_2 = O(\frac{1}{r}) .$$

Expanding $\log(1+z)$ in a power series shows

$$(1+z)\log(1+z) - z = \frac{1}{2} z^2 [1 + zg(z)] ,$$

where $g(z)$ is analytic for $|z| < 1$. If

$$w = z[1 + zg(z)]^{1/2}$$

where the square root is 1 for $z = 0$, then there is a $c > 0$ such that for $|z| \leq c$

$$z = w + w^2 g_1(w) ,$$

where $g_1(w)$ is real for real w and is analytic. Let

$$-c_1 = -c[1 - cg(-c)]^{1/2}, \; c_2 = c[1 + cg(c)]^{1/2} .$$

Then if $g_2(w) = 2g_1(w) + wg_1'(w)$,

$$I_2 = \int_{-c_1}^{c_2} e^{ir\,u^2/2}(1 + ug_2(u))du$$

$$= \int_{-c_1}^{c_2} e^{ir\,u^2/2} du + \int_{-c_1}^{c_2} e^{ir\,u^2/2} g_2(u) \, d(\frac{u^2}{2}) .$$

Integrating the second term by parts gives $O(\frac{1}{r})$. Hence

$$I_2 = \int_{-\infty}^{\infty} e^{ir\,u^2/2} du + J_1 + J_2 + O(\frac{1}{r}) ,$$

where

$$J_1 = \int_{c_2}^{\infty} e^{ir\,u^2/2} du = \int_{c_2}^{\infty} e^{ir\,u^2/2} u^{-1} d(\frac{u^2}{2})$$

and J_2 is similar. Integration by parts shows $J_1 = O(\frac{1}{r})$ and similarly for J_2. Hence

$$I_1 = \int_{-\infty}^{\infty} e^{iru^2/2}\, du + O(\frac{1}{r}) .$$

But an elementary change of contour allows the evaluation of the integral to give

$$I_1 = (\frac{2\pi}{r})^{1/2} e^{\pi i/4} + O(\frac{1}{r}) ,$$

which completes the proof. □

Lemma 1.2. For large A and $A < r \leq B \leq 2A$

(27) $$\int_A^B \exp[it \log \frac{t}{re}] (\frac{t}{2\pi})^{a-1/2} dt$$
$$= (2\pi)^{1-a} r^a e^{-ir+\pi i/4} + E(r,A,B) ,$$

where a is fixed and where

(28) $$E(r,A,B) = O(A^{a-1/2}) + O(\frac{A^{a+1/2}}{|A-r| + A^{1/2}}) + O(\frac{B^{a+1/2}}{|B-r| + B^{1/2}}) .$$

For $r \leq A$ or $r > B$,

$$\int_A^B \exp[it \log \frac{t}{re}] (\frac{t}{2\pi})^{a-1/2} dt = E(r,A,B,) .$$

Proof: Let $F(t,r) = \exp[it \log t/re]$. Let $A + A^{1/2} \leq r \leq B - B^{1/2}$. Then

$$\int_A^B F(t,r) (\frac{t}{2\pi})^{a-1/2} dt = \int_{r(1-c)}^{r(1+c)} F(t,r) (\frac{t}{2\pi})^{a-1/2} dt + J_1 + J_2 ,$$

where $J_1 = \int_{r(1-c)}^{A} F(t,r)(\frac{t}{2\pi})^{a-\frac{1}{2}}\, dt$ and J_2 is the integral over $(B, r(1+c))$. Since $\frac{d}{dt}(t \log \frac{t}{er}) = \log \frac{t}{r}$, integration by parts gives

$$(2\pi)^{a-\frac{1}{2}} J_1 = F(t,r) t^{a-\frac{1}{2}}/(\log \tfrac{t}{r}) i \Big]_{r(1-c)}^{A}$$

$$+ O(A^{a-\frac{1}{2}}) \int_{r(1-c)}^{A} dt/(t \log^2 \tfrac{r}{t})$$

$$+ O(A^{a-\frac{1}{2}}/(\log \tfrac{r}{A})) \int_{r(1-c)}^{A} dt/t$$

$$= O(A^{a-\frac{1}{2}}/(\log \tfrac{r}{A})) .$$

For $A + A^{1/2} \le r \le 2A$,

$$\log \frac{r}{A} \ge \frac{1}{4} \frac{r-A}{A} \ge \frac{1}{8} \frac{r-A}{A} + \frac{1}{8 A^{\frac{1}{2}}} ,$$

so $J_1 = E(r,A,B)$ and similarly $J_2 = E(r,A,B)$. By the previous lemma, for $A+A^{1/2} \le r \le B-B^{1/2}$ we have

$$\int_{r(1-c)}^{r(1+c)} F(t,r)(t/2\pi)^{a-\frac{1}{2}}\, dt = (2\pi)^{1-a} r^a e^{-ir+\pi i/4} + O(r^{a-\frac{1}{2}}) .$$

The error term is $O(A^{a-\frac{1}{2}}) = E(r,A,B)$. Thus (27) holds.

If $A-A^{\frac{1}{2}} < r < A+A^{\frac{1}{2}}$ then

$$\int_{A}^{B} F(t,r)(t/2\pi)^{a-\frac{1}{2}} dt = \int_{A}^{A+2A^{\frac{1}{2}}} + \int_{A+2A^{\frac{1}{2}}}^{B} F(t,r)(t/2\pi)^{a-\frac{1}{2}} dt$$

$$= O(A^a) + \int_{A+2A^{\frac{1}{2}}}^{B} F(t,r)(t/2\pi)^{a-\frac{1}{2}} dt$$

(where the second term does not appear if $B \leq A+2A^{1/2}$). The integral on the right is integrated by parts to give

$$O(A^a) + O(A^{a-\frac{1}{2}}/\log(A + \frac{2A^{\frac{1}{2}}}{r})) = O(A^a) .$$

Thus

$$\int_A^B F(t,r)\,(\frac{t}{2\pi})^{a-\frac{1}{2}}\,dt = O(A^a) = E(r,A,B)$$

for in the present range of r, $E(r,A,B) = O(A^a)$. Again the lemma is valid. The case $B-B^{1/2} < r < B+B^{1/2}$ is treated similarly.

If $r < A-A^{1/2}$, one integration by parts establishes the lemma directly where r is considered first in the range $r \leq \frac{3A}{4}$ and then $\frac{3A}{4} < r < A-A^{1/2}$. The case $r > B+B^{1/2}$ is treated similarly. □

Lemma 1.3. For $m = 1,2, \ldots$, A large, and $A < r \leq B \leq 2A$,

$$\int_A^B \exp[it \log \frac{t}{re}]\,(\frac{t}{2\pi})^{a-\frac{1}{2}}(\log \frac{t}{2\pi})^m\,dt$$

$$= (2\pi)^{1-a}\, r^a\, e^{-ir+\pi i/4}(\log \frac{r}{2\pi})^m + E(r,A,B)(\log A)^m,$$

while for $r \leq A$ or $r > B$,

$$\int_A^B \exp[it \log \frac{t}{re}]\,(\frac{t}{2\pi})^{a-\frac{1}{2}}(\log \frac{t}{2\pi})^m\,dt = E(r,A,B)(\log A)^m ,$$

where $E(r,A,B)$ is (28).

<u>Proof</u>: Using $F(t,r)$ as before, if $A-A^{1/2} < r < B+B^{1/2}$,

$$(29) \quad \int_A^B F(t,r)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log \frac{t}{2\pi}\right)^m dt$$

$$= \sum_{j=0}^{m} \binom{m}{j}\left(\log \frac{r}{2\pi}\right)^j \int_A^B F(t,r)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log \frac{t}{r}\right)^{m-j} dt .$$

Let $J_k = \int_A^B F(t,r)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log \frac{t}{r}\right)^k dt$ for $k \geq 1$. Then

$$(2\pi)^{a-\frac{1}{2}} J_k = \int_A^B \frac{t^{a-\frac{1}{2}}(\log t/r)^k}{i \log t/r} dF(t,r)$$

$$= -iF(t,r)t^{a-\frac{1}{2}}\left(\log \frac{t}{r}\right)^{k-1}\Big]_A^B$$

$$+ i \int_A^B F(t,r)t^{a-\frac{3}{2}}\left(\left(a-\frac{1}{2}\right)\left(\log \frac{t}{r}\right)^{k-1} + (k-1)\left(\log \frac{t}{r}\right)^{k-2}\right) dt$$

$$= O(A^{a-\frac{1}{2}}) .$$

(Note that for the above range of r, $\max_{t \in [A,B]} \left|\left(\log \frac{t}{r}\right)^k\right| = O(1)$.) The right-hand side of (29) is therefore

$$(30) \quad \left(\log \frac{r}{2\pi}\right)^m \int_A^B F(t,r)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} dt + O(A^{a-\frac{1}{2}}(\log A)^{m-1}) .$$

For $A < r \le B$ this is, by the previous lemma,

$$(\log \frac{r}{2\pi})^m\Big((2\pi)^{1-a}\, r^a\, e^{-ir+\pi i/4} + E(r,A,B)\Big)$$
$$+ (\log A)^{m-1}\, E(r,A,B)\ .$$

If $A-A^{1/2} < r \le A$ or $B < r < B+B^{1/2}$, we obtain from the previous lemma that (30) is

$$(\log \frac{r}{2\pi})^m\, E(r,A,B) + O(A^{a-1/2}(\log A)^{m-1})$$
$$= E(r,A,B)(\log A)^m\ .$$

This proves the lemma for $A-A^{1/2} < r < B+B^{1/2}$. Now suppose $r \le A-A^{1/2}$. We have

$$\int_A^B F(t,r)t^{a-1/2}(\log \frac{t}{2\pi})^m\, dt = \int_A^B t^{a-1/2}(\log \frac{t}{2\pi})^m \frac{dF(t,r)}{i\log t/r}$$

$$= \frac{F(t,r)t^{a-1/2}(\log t/2\pi)^m}{i\log t/r}\Bigg]_A^B - i\int_A^B \frac{F(t,r)t^{a-3/2}(\log t/2\pi)^m}{(\log t/r)^2}\, dt$$

$$+ i\int_A^B F(t,r)t^{a-3/2}\Big\{(a-\tfrac{1}{2})(\log \frac{t}{2\pi})^m + m(\log \frac{t}{2\pi})^{m-1}\Big\}\frac{dt}{\log t/r}$$

$$= O\Big(A^{a-1/2}(\log A)^m/(\log \frac{A}{r})\Big) + O(A^{a-1/2}(\log A)^m)\int_A^B \frac{dt}{t(\log t/r)^2}$$
$$+ O(A^{a-1/2}(\log A)^m/(\log \frac{A}{r}))\int_A^B \frac{dt}{t}$$

$$= O\Big(A^{a-1/2}(\log A)^m/(\log \frac{A}{r})\Big) = E(r,A,B)(\log A)^m\ .$$

The case $r \geq B+B^{1/2}$ is treated similarly. □

Lemma 1.4. Let $E(r,A,B)$ be as in (28), where A is large and $A < B \leq 2A$. Assume $\{b_n\}_{n=1}^{\infty}$ is a sequence of complex numbers such that $b_n \ll n^{\varepsilon}$ for any $\varepsilon > 0$. Then if $a > 1$,

$$\sum_{n=1}^{\infty} \frac{b_n}{n^a} E(2\pi n,A,B) \ll A^{a-\frac{1}{2}} .$$

Proof: Choose ε so that $0 < \varepsilon < a-1$. By (28)

$$\sum_{n=1}^{\infty} \frac{b_n}{n^a} E(2\pi n,A,B) \ll \sum_{n=1}^{\infty} n^{-a+\varepsilon} E(2\pi n,A,B)$$

$$\ll A^{a-\frac{1}{2}} \sum_{n=1}^{\infty} n^{-a+\varepsilon} + A^{a+\frac{1}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{a-\varepsilon}(|A-2\pi n| + A^{1/2})}$$

$$+ B^{a+\frac{1}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{a-\varepsilon}(|B-2\pi n| + B^{1/2})}$$

Since $\sum_{n=1}^{\infty} n^{-a+\varepsilon} \ll 1$, it evidently suffices to show that for large C

$$\sum_{n=1}^{\infty} \frac{1}{n^{a-\varepsilon}(|C-2\pi n| + C^{1/2})} \ll C^{-1} .$$

We write this sum as

$$S_1 + S_2 + S_3 + S_4 + S_5 ,$$

where S_1 is over $[1, \frac{C}{2\pi})$, S_2 is over $[\frac{C}{2\pi}, (C - C^{1/2})/2\pi)$,

S_3 is over $[(C - C^{1/2})/2\pi, (C + C^{1/2})/2\pi)$, S_4 is over $[(C + C^{1/2})/2\pi, C/\pi)$, and S_5 is over $[C/\pi, \infty)$. We easily find that

$$S_1 \ll C^{-1} \sum_{n<C/2\pi} n^{-a+\varepsilon} \ll C^{-1} ,$$

$$S_2 \ll C^{-a+\varepsilon} \sum_{C^{1/2}<n\leq C/2} n^{-1} \ll C^{-1} ,$$

$$S_3 \ll C^{-1/2} \sum_{(C-C^{1/2})/2\pi\leq n<(C+C^{1/2})/2\pi} n^{-a+\varepsilon} \ll C^{-a+\varepsilon} \ll C^{-1} ,$$

$$S_4 \ll C^{-a+\varepsilon} \sum_{C^{1/2}\leq n<C} n^{-1} \ll C^{-1} ,$$

and finally that

$$S_5 \ll C^{-1} \sum_{C/\pi\leq n} n^{-a+\varepsilon} \ll C^{-1} .$$

This gives the result. □

Lemma 1.5. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that for any $\varepsilon > 0$, $b_n \ll n^{\varepsilon}$. Let $a > 1$ and let m be a non-negative integer. Then for T sufficiently large,

$$(31) \quad \frac{1}{2\pi} \int_1^T \left(\sum_{n=1}^{\infty} b_n \, n^{-a-it} \right) \chi(1-a-it) \left(\log \frac{t}{2\pi}\right)^m dt$$

$$= \sum_{1\leq n\leq T/2\pi} b_n (\log n)^m + O(T^{a-1/2} (\log T)^m) .$$

Proof: By (11) we have

$$(32)\quad \frac{1}{2\pi}\int_{T/2}^{T}\left(\sum_{n=1}^{\infty} b_n\, n^{-a-it}\right)\chi(1-a-it)\left(\log\frac{t}{2\pi}\right)^m dt$$

$$= \frac{1}{2\pi}\int_{T/2}^{T}\left(\sum_{n=1}^{\infty} b_n\, n^{-a-it}\right) e^{-\pi i/4}$$

$$\cdot\ \exp\left[it\log\frac{t}{2\pi e}\right]\left(\frac{t}{2\pi}\right)^{a-1/2}\left(\log\frac{t}{2\pi}\right)^m dt$$

$$+\ O\left(\int_{T/2}^{T}\left(\sum_{n=1}^{\infty}|b_n|n^{-a}\right)t^{a-3/2}(\log t)^m\ dt\right).$$

Since $b_n \ll n^{\varepsilon}$, $\sum_{n=1}^{\infty} |b_n|n^{-a} \ll 1$ if $a > 1$. The error term in (32) is therefore

$$(33)\qquad \ll \int_{T/2}^{T} t^{a-3/2}(\log t)^m\ dt \ll T^{a-1/2}(\log T)^m\ .$$

To treat the main term on the right-hand side of (32) we write it as

$$(34)\quad \sum_{n=1}^{\infty} b_n\, n^{-a}\, e^{-\pi i/4}$$

$$\cdot\left(\frac{1}{2\pi}\int_{T/2}^{T}\exp\left[it\log\frac{t}{2\pi}\,ne\right]\left(\frac{t}{2\pi}\right)^{a-1/2}\left(\log\frac{t}{2\pi}\right)^m dt\right),$$

the inversion of summation and integration being justified by absolute convergence. Now the integral in (34) is of the form estimable by Lemma 1.3 with $A = \frac{T}{2}$, $B = T$, and

$r = 2\pi n$. Thus (34) is equal to

$$\sum_{T/4\pi<n\leq T/2\pi} b_n(\log n)^m + (\log \frac{T}{2})^m \sum_{n=1}^{\infty} b_n n^{-a}E(2\pi n, \frac{T}{2}, T)$$

for large T . By Lemma 1.4 the second term is

$$\ll (\log \frac{T}{2})^m (\frac{T}{2})^{a-1/2} \ll T^{a-1/2}(\log T)^m .$$

Hence (34) is equal to

$$\sum_{T/4\pi<n\leq T/2\pi} b_n(\log n)^m + O(T^{a-1/2}(\log T)^m) .$$

Using this and (33) in (32) we obtain

(35) $$\frac{1}{2\pi}\int_{T/2}^{T}\left(\sum_{n=1}^{\infty} b_n n^{-a-it}\right)\chi(1-a-it)(\log \frac{t}{2\pi})^m\, dt$$
$$= \sum_{T/4\pi<n\leq T/2\pi} b_n(\log n)^m + O(T^{a-1/2}(\log T)^m)$$

for $T \geq T_0$, say. Now let ℓ be the unique integer such that $T_0 \leq \frac{T}{2^\ell} < 2T_0$. We add together the result of (35) for the ranges $[\frac{T}{2^\ell}, \frac{T}{2^{\ell-1}}]$, $[\frac{T}{2^{\ell-1}}, \frac{T}{2^{\ell-2}}]$, ..., $[\frac{T}{2}, T]$ and obtain

$$\frac{1}{2\pi}\int_{T/2^\ell}^{T}\left(\sum_{n=1}^{\infty} b_n n^{-a-it}\right)\chi(1-a-it)(\log \frac{t}{2\pi})^m\, dt$$
$$= \sum_{1/2\pi\ T/2^\ell<n\leq T/2\pi} b_n(\log n)^m + O(T^{a-1/2})\log T)^m) .$$

Proof: The case $\nu = 0$ is trivial. For $\nu = 1$ we write

(36) $$\chi^{(1)}(1-s) = \chi(1-s) \cdot \frac{\chi'}{\chi}(1-s)$$

i.e., $\chi(1-s) \cdot \frac{\chi'}{\chi}(1-s)$. The logarithmic derivative of (7) is

$$\frac{\chi'}{\chi}(1-s) = \log \pi - \frac{1}{2}\,\psi(\frac{s}{2}) - \frac{1}{2}\,\psi(\frac{(1-s)}{2})$$

and by (16) this is

(37) $$\frac{\chi'}{\chi}(1-s) = \log \pi - \frac{1}{2}\log\frac{s}{2} - \frac{1}{2}\log\frac{(1-s)}{2} + O(\frac{1}{|s|}) + O(\frac{1}{|s-1|}) ,$$

or

(38) $$\frac{\chi'}{\chi}(1-s) = -\log\frac{|t|}{2\pi} + O(\frac{1}{|t|}) \qquad (|t| \geq 1) .$$

Also, by (11),

(39) $$\chi(1-s) = O(|t|^{\sigma-1/2}) \qquad (|t| \geq 1) .$$

Combining (36), (38), and (39) yields

$$\chi^{(1)}(1-s) = \chi(1-s)(-\log\frac{|t|}{2\pi} + O(\frac{1}{|t|}))$$
$$= \chi(1-s)(-\log\frac{|t|}{2\pi}) + O(|t|^{\sigma-3/2})$$

for $|t| \geq 1$. This proves the lemma when $\nu = 1$.

Now assume the lemma is true for $\nu < \mu$. Differentiating (36) $\mu-1$ times and using Leibniz's rule leads to

$$(40) \quad \chi^{(\mu)}(1-s) = \sum_{\nu=0}^{\mu-1} \binom{\mu-1}{\nu} \chi^{(\nu)}(1-s)\left(\frac{\chi'}{\chi}(1-s)\right)^{(\mu-\nu-1)} .$$

By (37) and Cauchy's inequality for analytic functions applied to a small disc centered at s we find

$$\frac{d^k}{dw^k}\left(\frac{\chi'}{\chi}(1-w) - \log \pi + \frac{1}{2}\log \frac{w}{2} + \frac{1}{2}\log\left(\frac{(1-w)}{2}\right)\right)\Big|_{w=s} \ll \frac{1}{|t|} .$$

If $k \geq 1$ this gives

$$(41) \qquad \left(\frac{\chi'}{\chi}(1-s)\right)^{(k)} \ll \frac{1}{|t|} .$$

From (39), (40), (41), and the inductive hypothesis it follows that

$$\chi^{(\mu)}(1-s) = \chi^{(\mu-1)}(1-s)\ \frac{\chi'}{\chi}(1-s) + \sum_{\nu=0}^{\mu-2} \binom{\mu-1}{\nu}\Bigg(\chi(1-s)\left(-\log \frac{|t|}{2\pi}\right)^{\nu}$$

$$+ O(|t|^{\sigma-3/2}(\log |t|)^{\nu-1})\Bigg) O\left(\frac{1}{|t|}\right)$$

$$= \chi^{(\mu-1)}(1-s)\frac{\chi'}{\chi}(1-s) + O(|t|^{\sigma-3/2}(\log |t|)^{\mu-2})$$

$$= \left(\chi(1-s)\left(-\log \frac{|t|}{2\pi}\right)^{\mu-1} + O(t^{\sigma-3/2}(\log |t|)^{\mu-2})\right)\frac{\chi'}{\chi}(1-s)$$

$$+ O(|t|^{\sigma-3/2}(\log |t|)^{\mu-2})$$

$$= \chi(1-s)(-\log \frac{|t|}{2\pi})^{\mu-1} \frac{\chi'}{\chi}(1-s) + O(|t|^{\sigma-3/2}(\log |t|)^{\mu-2})$$

By (38) the last line is

$$= \chi(1-s)(-\log \frac{|t|}{2\pi})^{\mu-1}(-\log \frac{|t|}{2\pi} + O(\frac{1}{|t|}))$$

$$+ O(|t|^{\sigma-3/2}(\log |t|)^{\mu-2}$$

$$= \chi(1-s)(-\log \frac{|t|}{2\pi})^{\mu} + O(|t|^{\sigma-3/2}(\log |t|)^{\mu-1}) .$$

This completes the proof. □

Lemma 1.7. Let $f(w) = \sum_{n=1}^{\infty} \frac{a_n}{n^w}$ $(\mathrm{Re}\, w > 1)$, where $a_n \ll \phi(n)$, $\phi(n)$ being a non-decreasing function, and suppose

$$\sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} = O(\frac{1}{(\sigma-1)^\alpha}) \qquad \text{as } \sigma \to 1 .$$

Then if $c > 0$, $\sigma+c > 1$, x is not an integer, and N is the integer nearest x,

$$\sum_{n \leq x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) \frac{x^w}{w} dw + O(\frac{x^c}{T(\sigma+c-1)^\alpha})$$

$$+ O(\frac{\phi(2x)x^{1-\sigma} \log x}{T}) + O(\frac{\phi(N)x^{1-\sigma}}{T|x-N|}) .$$

For a proof of this lemma see Titchmarsh [19; pp. 53-55].

Lemma 1.8. Let $\zeta^{(\mu)}(s)\zeta^{(\nu)}(s) = \sum_{n=1}^{\infty} \frac{A_n}{n^s}$ (Re $s > 1$) , where $\mu,\nu \geq 0$. Then

$$\sum_{n\leq x} A_n = \frac{(-1)^{\mu+\nu}\,\mu!\,\nu!}{(\mu+\nu+1)!}\,x(\log x)^{\mu+\nu+1} + O(x(\log x)^{\mu+\nu}) . \tag{42}$$

Proof: First we assume x is half an odd integer and apply Lemma 1.7 with $f(w) = \zeta^{(\mu)}(w)\zeta^{(\nu)}(w)$, $a_n = A_n$, $\sigma = 0$, and $c > 1$. Since f has a pole of order $\mu+\nu+2$ at $w = 1$, we must take $\alpha = \mu+\nu+2$. Also if $d(n)$ denotes the divisor function, we have

$$|A_n| = \sum_{d|n} (\log d)^{\mu}(\log \tfrac{n}{d})^{\nu} \ll (\log n)^{\mu+\nu}\, d(n) \ll n^{\varepsilon}$$

for any $\varepsilon > 0$. Hence we may choose $\phi(n) = n^{\varepsilon}$ in Lemma 1.7. We then obtain

$$\sum_{n\leq x} A_n = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\,\frac{x^s}{s}\,ds + O\left(\frac{x^c}{T(c-1)^{\mu+\nu+2}}\right)$$
$$+ O\left(\frac{x^{1+\varepsilon}}{T}\right) .$$

Moving the contour to the left as far as $s = \frac{1}{2}$ we find

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\,\frac{x^s}{s}\,ds$$

$$= -\frac{1}{2\pi i}\left(\int_{c+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} + \int_{\frac{1}{2}-iT}^{c-iT}\right)\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\,\frac{x^s}{s}\,ds$$

$$+ \operatorname{Res}\ \zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\,\frac{x^s}{s}\Big]_{s=1} .$$

Since $\zeta^{(\nu)}(s) = \frac{(-1)^{\nu}\,\nu!}{(s-1)^{\nu+1}} + c_0 + \ldots$ for s near 1, the residue is easily seen to be

$$\frac{(-1)^{\mu+\nu}\mu!\,\nu!}{(\mu+\nu+1)!}\,x\,P_{\mu+\nu+1}(\log x) ,$$

where $P_{\mu+\nu+1}$ is a monic polynomial of degree $\mu+\nu+1$. By (21) we see that

$$\int_{\frac{1}{2}+iT}^{c+iT} \zeta^{(\mu)}(\sigma+iT)\zeta^{(\nu)}(\sigma+iT)\,\frac{x^s}{s}\,ds$$

$$\ll \max_{\frac{1}{2}\le\sigma\le c} |\zeta^{(\mu)}(\sigma+iT)\zeta^{(\nu)}(\sigma+iT)|\,\frac{x^\sigma}{T}$$

$$\ll T^{\frac{1}{2}+\varepsilon}\,\frac{x^c}{T} = \frac{x^c}{T^{\frac{1}{2}-\varepsilon}} .$$

The same bound holds for the integral over $[\frac{1}{2} - iT,\ c-iT]$. The remaining integral is (also by (21))

$$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \zeta^{(\mu)}(s)\zeta^{(\nu)}(s)\,\frac{x^s}{s}\,ds \ll x^{1/2}\int_1^T t^{\frac{1}{2}+\varepsilon}\,\frac{dt}{t} \ll x^{1/2}\,T^{\frac{1}{2}+\varepsilon}.$$

Combining these estimates we obtain

$$\sum_{n\leq x} A_n = \frac{(-1)^{\mu+\nu}\mu!\nu!}{(\mu+\nu+1)!} x P_{\mu+\nu+1}(\log x) + O\left(\frac{x^c}{T(c-1)^{\mu+\nu+2}}\right)$$

$$+ O\left(\frac{x^{1+\varepsilon}}{T}\right) + O\left(\frac{x^c}{T^{\frac{1}{2}-\varepsilon}}\right) + O(x^{1/2}\, T^{\frac{1}{2}+\varepsilon}) .$$

Now taking $c = 1+\varepsilon$ and $T = x^{1/2}$, the errors are all $O(x^{\frac{3}{4}+\frac{3}{2}\varepsilon})$. This proves the lemma when x is half an odd integer. Now varying x in (42) by $O(1)$ changes the left-hand side of (42) by at most $O(x^{\varepsilon})$ and the main term on the right-hand side by at most $O((\log x)^{\mu+\nu+1})$. As both these variations can be absorbed in the error term in (42), we see that (42) holds for all x. This completes the proof. □

Lemma 1.9. Let $\frac{\zeta'}{\zeta}(s)\zeta^{(\mu)}(s)\zeta^{(\nu)}(s) = \sum_{n=1}^{\infty} \frac{B_n}{n^s}$ $(\mathrm{Re}\, s > 1)$. If $\mu,\nu \geq 0$, then

$$\sum_{n\leq x} B_n = \frac{(-1)^{\mu+\nu+1}\mu!\nu!}{(\mu+\nu+2)!} x(\log x)^{\mu+\nu+2}\left(1 + O\left(\frac{1}{\log x}\right)\right) .$$

Proof: As in Lemma 1.8, we write A_n for the n^{th} coefficient of the Dirichlet series for $\zeta^{(\mu)}(s)\zeta^{(\nu)}(s)$ and we set

$$S(x) = \sum_{n\leq x} A_n .$$

Also we write

$$\psi(x) = \sum_{n \le x} \Lambda(n) ,$$

where $\Lambda(n)$ is the n^{th} coefficient of the Dirichlet series for $-\frac{\zeta'}{\zeta}(s)$. (There should be no confusion with the Euler psi-function in the present context.) Then

$$\sum_{n \le x} B_n = - \sum_{n \le x} \sum_{d|n} \Lambda(d) A_{n/d} = - \sum_{d \le x} \Lambda(d) \sum_{e \le x/d} A_e$$

$$= - \sum_{d \le x} \Lambda(d) S(\tfrac{x}{d}) .$$

By Lemma 1.8 the last sum is

$$= \sum_{d \le x} \Lambda(d) \left(\frac{(-1)^{\mu+\nu+1}}{(\mu+\nu+1)!} \mu!\nu! \frac{x}{d} ((\log \tfrac{x}{d})^{\mu+\nu+1} + O((\log \tfrac{x}{d})^{\mu+\nu})) \right)$$

$$= \frac{(-1)^{\mu+\nu+1} \mu!\nu!}{(\mu+\nu+1)!} x \sum_{d \le x} \frac{\Lambda(d)}{d} (\log \tfrac{x}{d})^{\mu+\nu+1}$$

$$+ O(x \sum_{d \le x} \frac{\Lambda(d)}{d} (\log \tfrac{x}{d})^{\mu+\nu}) .$$

It is now not difficult to see that the lemma will follow if we prove that for $k \ge 0$,

$$\sum_{d \le x} \frac{\Lambda(d)}{d} (\log \tfrac{x}{d})^k = \frac{1}{k+1} (\log x)^{k+1} + O((\log x)^k) .$$

To this end, we write $\psi(u) = u + E(u)$ where, by the prime number theorem, $E(u) = O(u \exp(-c\sqrt{\log u}))$ for some $c > 0$. Then

$$\sum_{d\leq x} \frac{\Lambda(d)}{d}(\log \frac{x}{d})^k = \int_1^x \frac{(\log x/u)^k}{u}\, d\psi(u)$$

$$= \int_1^x \frac{(\log x/u)^k}{u}\, du + \int_1^x \frac{(\log x/u)^k}{u}\, dE(u)$$

$$= \frac{1}{k+1}(\log x)^{k+1} + \frac{E(u)(\log x/u)^k}{u}\Big]_1^x$$

$$+ \int_1^x \frac{E(u)}{u^2}(k(\log \frac{x}{u})^{k-1} + (\log \frac{x}{u})^k)\, du$$

$$= \frac{1}{k+1}(\log x)^{k+1} + O((\log x)^k)$$

$$+ O\left(\int_1^x \frac{|E(u)|}{u^2}(\log \frac{x}{u})^k\, du\right).$$

Now

$$\int_1^x \frac{|E(u)|}{u^2}(\log \frac{x}{u})^k\, du \ll \int_1^x \exp(-c\sqrt{\log u})(\log \frac{x}{u})^k \frac{du}{u}$$

$$\ll (\log x)^k \int_1^{x/2} \exp(-c\sqrt{\log u})\frac{du}{u}$$

$$+ \exp(-c\sqrt{\log x/2}) \int_{x/2}^x (\log \frac{x}{u})^k \frac{du}{u}$$

$$\ll (\log x)^k + \exp(-c\sqrt{\log x/2}) \ll (\log x)^k .$$

This gives the required result. □

Lemma 1.10. Suppose $\delta \neq 0$ is real and $\zeta^2(s)\frac{\zeta'}{\zeta}(s-i\delta) = \sum_{n=1}^{\infty} \frac{C_n(i\delta)}{n^s}$ (Re $s > 1$) . Then

$$\sum_{n \leq x} C_n(i\delta) = x\left(\frac{\log x}{i\delta} + \frac{x^{i\delta} - 1}{\delta^2}\right) + O(x \log x)$$

where the implicit constant in the O-term is independent of δ .

Proof: Since for Re $s > 1$ we have $\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}$ and $\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} - \frac{\Lambda(n)}{n^s}$, we see that

$$\sum_{n \leq x} C_n(i\delta) = - \sum_{n \leq x} \sum_{d | n} \Lambda(d) d^{i\delta} \, d(\tfrac{n}{d})$$

$$= - \sum_{d \leq x} \Lambda(d) d^{i\delta} \sum_{e \leq x/d} d(e) \, .$$

We use the well-known formula

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(x^{1/2})$$

to see that

(43) $$\sum_{n \leq x} C_n(i\delta) = - x \sum_{d \leq x} \frac{\Lambda(d)}{d^{1-i\delta}} \log \frac{x}{d} - (2\gamma-1)x \sum_{d \leq x} \frac{\Lambda(d)}{d^{1-i\delta}} + O\left(x^{1/2} \sum_{d \leq x} \frac{\Lambda(d)}{d^{1/2}}\right) .$$

We write $\psi(u) = u + E(u)$ as in the proof of Lemma 1.9, where $E(u) = (u \exp(-c\sqrt{\log u}))$. Then the error term in (43) is

$$\ll x^{1/2} \int_1^x \frac{d\psi(u)}{u^{1/2}} = x^{1/2}\left(\frac{\psi(x)}{x^{1/2}} + \frac{1}{2}\int_1^x \frac{\psi(u)\,du}{u^{3/2}}\right)$$

$$\ll \psi(x) + x^{1/2}\int_1^x \frac{du}{u^{1/2}} \ll x \quad .$$

Furthermore

$$\sum_{d\leq x} \frac{\Lambda(d)}{d^{1-i\delta}} \log\frac{x}{d} = \int_1^x \frac{\log x/u}{u^{1-i\delta}}\, d\psi(u)$$

$$= \int_1^x \frac{\log x/u}{u^{1-i\delta}}\, du + \int_1^x \frac{\log x/u}{u^{1-i\delta}}\, d\,E(u)$$

$$= -\frac{\log x}{i\delta} + \frac{1}{i\delta}\int_1^x \frac{du}{u^{1-i\delta}} + \frac{E(u)\log x/u}{u^{1-i\delta}}\Bigg]_1^x - \int_1^x \frac{E(u)}{u^{2-i\delta}}\left(1 + (1-i\delta)\log\frac{x}{u}\right)$$

$$= -\frac{\log x}{i\delta} + \frac{1-x^{i\delta}}{\delta^2} + E(1)\log x + + O\left(\log x \int_1^x \frac{\exp(-c\sqrt{\log u})}{u}\, du\right)$$

$$= -\frac{\log x}{i\delta} + \frac{1-x^{i\delta}}{\delta^2} + O(\log x) \quad .$$

Also the second sum on the right-hand side of (43) is

$$\sum_{d\leq x} \frac{\Lambda(d)}{d^{1-i\delta}} = \int_1^x \frac{d\psi(u)}{u^{1-i\delta}} = \int_1^x \frac{du}{u^{1-i\delta}} + \int_1^x \frac{dE(u)}{u^{1-i\delta}}$$

$$= \frac{u^{i\delta}}{i\delta}\Big|_1^x + \frac{E(u)}{u^{1-i\delta}}\Big|_1^x + (1-i\delta)\int_1^x \frac{E(u)du}{u^{2-i\delta}}$$

$$= \frac{x^{i\delta}-1}{i\delta} + O(1) = O(\log x) \ .$$

Substituting these estimates into (43) we see that

$$\sum_{n\leq x} C_n(i\delta) = -x(-\frac{\log x}{i\delta} + \frac{1-x^{i\delta}}{\delta^2} + O(\log x)) + O(x)$$

$$= x(\frac{\log x}{i\delta} + \frac{x^{i\delta}-1}{\delta^2}) + O(x \log x) \ .$$

This proves the lemma. □

Lemma 1.11. Suppose that

$$\sum_{n\leq x} a_n = x(\log x)^k + O(x(\log x)^{k-1})$$

for some fixed $k \geq 1$. If $\ell \geq 0$ is fixed then

$$\sum_{n\leq x} a_n(\log n)^\ell = x(\log x)^{k+\ell} + O(x(\log x)^{k+\ell-1}) \ .$$

Proof: Write $S(x) = \sum_{n \le x} a_n$. Then

$$\sum_{n \le x} a_n (\log n)^{\ell} = \int_{1^-}^{x} (\log u)^{\ell} \, dS(u)$$

$$= S(u)(\log u)^{\ell} \Big|_{1^-}^{x} - \ell \int_{1}^{x} S(u)(\log u)^{\ell-1} \frac{du}{u}$$

$$= S(x)(\log x)^{\ell} - \ell \int_{1}^{x} (\log u)^{k+\ell-1} \, du$$

$$+ O\left(\int_{1}^{x} (\log u)^{k+\ell-2} \, du\right)$$

$$= S(x)(\log x)^{\ell} + O(x(\log x)^{k+\ell-1})$$

$$= x(\log x)^{k+\ell} + O(x(\log x)^{k+\ell-1}) . \qquad \square$$

§5. Completion of the Proof of Theorem 1.1

We now have the lemmas necessary for completing the proofs of Theorems 1.1 and 1.2 at our disposal. In this section we prove Theorem 1.1.

By (23) and (26) we know that

$$(44) \qquad \sum_{1 \le \gamma \le T} \zeta(\rho+i\delta)\zeta(1-\rho-i\delta)$$

$$= 2 \operatorname{Re} I_1(0,0,i\delta) + O(T^{a-\frac{1}{2}+\varepsilon}) ,$$

and upon replacing s by $a+it$ and using (16) to estimate $\psi(\frac{s}{2})$, we get

$$(46) \qquad \frac{\xi'}{\xi}(a+it) = \frac{\zeta'}{\zeta}(a+it) + \frac{1}{2}\log\frac{t}{2\pi} + \frac{\pi i}{4} + O(\frac{1}{|t|}) ,$$

provided that $|a+it| \geq 1$. (The condition $|a+it| \geq 1$ arises from the hypothesis $|s| \geq \frac{1}{2}$ in (16).) As $|a+it| \geq 1$ when $t \in [1, T+\delta]$, we may substitute the right-hand side of (46) into the integrand in (45). This yields

$$I_1(0,0,i\delta) = \frac{1}{2\pi}\int_1^{T+\delta}\frac{\zeta'}{\zeta}(a+it-i\delta)\zeta^2(a+it)\chi(1-a-it)dt$$

$$+ \frac{1}{2\pi}\int_1^{T+\delta} \frac{1}{2}\zeta^2(a+it)\chi(1-a-it)\log\frac{t}{2\pi}dt$$

$$+ \frac{1}{2\pi}\int_1^{T+\delta} \frac{\pi i}{4}\zeta^2(a+it)\chi(1-a-it)dt$$

$$+ O\left(\int_1^{T+\delta}|\zeta(a+it)|^2 \ |\chi(1-a-it)|\frac{dt}{t}\right) + O(1) .$$

By the estimates available from (11) and (21), the next-to-last error term is

$$\ll \int_1^{T+\delta} t^{a-3/2+\varepsilon}dt \ll T^{a-1/2+\varepsilon} .$$

Therefore we may write

$$(47) \qquad I_1(0,0,i\delta) = I_{11} + I_{12} + I_{13} + O(T^{a-\frac{1}{2}+\varepsilon}) ,$$

where

$$I_{11} = \frac{1}{2\pi} \int_1^{T+\delta} \frac{\zeta'}{\zeta}(a+it-i\delta)\zeta^2(a+it)\chi(1-a-it)\,dt ,$$

$$I_{12} = \frac{1}{2\pi} \int_1^{T+\delta} \frac{1}{2} \zeta^2(a+it)\chi(1-a-it) \log \frac{t}{2\pi}\,dt ,$$

and

$$I_{13} = \frac{1}{2\pi} \int_1^{T+\delta} \frac{\pi i}{4} \zeta^2(a+it)\chi(1-a-it)\,dt .$$

To treat I_{11} we write $\frac{\zeta'}{\zeta}(s-i\delta)\zeta^2(s) = \sum_{n=1}^{\infty} \frac{C_n(i\delta)}{n^s}$ (Re $s > 1$) so that

$$I_{11} = \frac{1}{2\pi} \int_1^{T+\delta} \left(\sum_{n=1}^{\infty} C_n(i\delta) n^{-a-it} \right) \chi(1-a-it)\,dt .$$

This is precisely the type of integral Lemma 1.5 estimates. We need to check, however, that $C_n(i\delta) \ll n^{\varepsilon}$ for any $\varepsilon > 0$. But

$$C_n(i\delta) = \sum_{d|n} \Lambda(d) d^{i\delta}\, d(\tfrac{n}{d}) \ll \sum_{d|n} \Lambda(d) (\tfrac{n}{d})^{\varepsilon/2}$$

$$\ll n^{\varepsilon/2} \sum_{d|n} \Lambda(d) = n^{\varepsilon/2} \log n \ll n^{\varepsilon} .$$

Hence we may apply Lemma 1.5. We obtain

$$I_{11} = \sum_{1 \le n \le (T+\delta)/2\pi} C_n(i\delta) + O(T^{a-1/2})$$

$$= \sum_{1 \le n \le T/2\pi} C_n(i\delta) + O(T^{a-1/2}) .$$

Finally, using Lemma 1.10 to estimate the sum gives

$$(48) \qquad I_{11} = \frac{T}{2\pi}\left(\frac{\log T/2\pi}{i\delta} + \frac{(T/2\pi)^{i\delta} - 1}{\delta^2}\right) + O(T \log T) + O(T^{a-1/2}) .$$

We now estimate I_{12} . Write $\zeta^2(s) = \sum_{n=1}^{\infty} d(n)n^{-s}$ (Re s > 1) . Then

$$I_{12} = \frac{1}{2\pi}\int_1^{T+\delta}\left(\sum_{n=1}^{\infty} \frac{1}{2}\, d(n)n^{-a-it}\right)\chi(1-a-it)\log\frac{t}{2\pi}\, dt .$$

Since $d(n) \ll n^{\varepsilon}$, we have by Lemma 1.5,

$$I_{12} = \frac{1}{2}\sum_{1 \le n \le (T+\delta)/2\pi} d(n)\log n + O(T^{a-1/2}\log T)$$

$$= \frac{1}{2}\sum_{1 \le n \le T/2\pi} d(n)\log n + O(T^{a-1/2}\log T) .$$

From the elementary fact that

$$(49) \qquad \sum_{n \le x} d(n) = x \log x + O(x)$$

and Lemma 1.11, it follows that

$$\sum_{n\le x} d(n)\log n = x(\log x)^2 + O(x \log x) .$$

Thus

(50) $$I_{12} = \frac{T}{4\pi}\left(\log \frac{T}{2\pi}\right)^2 + O(T \log T) + O(T^{a-1/2} \log T) .$$

We treat I_{13} in a manner analogous to I_{12} . Clearly

$$I_{13} = \frac{1}{2\pi}\int_1^{T+\delta} \frac{\pi i}{4}\left(\sum_{n=1}^{\infty} d(n)n^{-a-it}\right)\chi(1-a-it)dt .$$

Applying Lemma 1.5 and (49) to this yields

(51) $$I_{13} = \frac{\pi i}{4}\sum_{1\le n\le (T+\delta)/2\pi} d(n) + O(T^{a-1/2})$$

$$= O(T \log T) + O(T^{a-1/2}) .$$

Combining (47), (48), (50), and (51), and taking $a = \frac{5}{4}$, we find that

$$I_1(0,0,i\delta) = \frac{T}{2\pi}\left(\frac{\log T/2\pi}{i\delta} + \frac{(T/2\pi)^{i\delta} - 1}{\delta^2}\right)$$

$$+ \frac{T}{4\pi}\left(\log \frac{T}{2\pi}\right)^2 + O(T \log T) .$$

Inserting this into (44) and setting $a = \frac{5}{4}$ gives

(52) $$\sum_{1\le\gamma\le T} \zeta(\rho+i\delta)\zeta(1-\rho-i\delta) = \frac{T}{2\pi}\left(\log \frac{T}{2\pi}\right)^2$$

$$+ \frac{T}{\pi}\left(\frac{\cos(\delta \log T/2\pi) - 1}{\delta^2}\right) + O(T \log T) ,$$

where the error term is independent of δ . Now take

$\delta = \frac{2\pi c}{(\log T/2\pi)}$ with $c \neq 0$, c real, and $|c| \le \frac{(\log T/2\pi)}{4}$.

The right-hand side of (52) is then

$$= \frac{T}{2\pi}(\log T)^2 + \frac{T}{\pi}\left(\frac{\cos 2\pi c - 1}{4\pi^2 c^2} (\log T)^2\right) + O(T \log T)$$

$$= \frac{T}{2\pi}(\log T)^2\left(1 - \left(\frac{\sin \pi c}{c}\right)^2\right) + O(T \log T) ,$$

where the error is uniform in c . Thus for T large and $T \in S$,

(53) $$\sum_{1\le\gamma\le T} \zeta\left(\rho+ \frac{2\pi i c}{\log T/2\pi}\right)\zeta\left(1-\rho- \frac{2\pi i c}{\log T/2\pi}\right)$$

$$= \left(1 - \left(\frac{\sin \pi c}{\pi c}\right)^2\right) \frac{T}{2\pi} (\log T)^2 + O(T \log T) .$$

By our construction of S , any large real number is within $O(1)$ of an element of S . Furthermore, there are at most $O(\log T)$ zeros of $\zeta(s)$ within a distance $O(1)$ of $\frac{1}{2} + iT$. Thus by (21), a change in T of order $O(1)$ induces a change in the left-hand side of (53) of size at most $O(T^{1/2+\varepsilon} \log T)$. On the other hand, a change of $O(1)$ in T in the main term on the right-hand side of (53) induces a change of at most $O((\log T)^2)$. As both these variations are smaller than the error term in (53), (53) is established for all large T . This is the first part of Theorem 1.1.

Finally, to prove (4) in Theorem 1.1 it suffices to note that the Riemann hypothesis and reflection principle imply that each summand on the left-hand side of (3) is

$$|\zeta(\tfrac{1}{2} + i\gamma + \frac{2\pi i c}{\log T/2\pi})|^2 . \qquad \square$$

§6. Completion of the Proof of Theorem 1.2.

By (24) and (26) we have

$$(54) \qquad \sum_{1\le\gamma\le T} \zeta^{(\mu)}(\rho)\zeta^{(\nu)}(1-\rho) = I_1(\mu,\nu,0) + \overline{I_1(\nu,\mu,0)} + O(T^{a-\frac{1}{2}+\varepsilon})$$

where $\mu,\nu \ge 1$, $a > 1$, T is large, $T \varepsilon S$, and

$$I_1(\mu,\nu,0) = \frac{1}{2\pi i}\int_{a+i}^{a+iT} \frac{\xi'}{\xi}(s)\zeta^{(\mu)}(s)\zeta^{(\nu)}(1-s)\,ds .$$

Differentiating (12) according to Leibniz's rule yields

$$\zeta^{(\nu)}(1-s) = \sum_{\kappa=0}^{\nu} \binom{\nu}{\kappa}(-1)^\kappa \zeta^{(\kappa)}(s)\chi^{(\nu-\kappa)}(1-s) .$$

We substitute this into the formula for $I_1(\mu,\nu,0)$ and obtain

$$I_1(\mu,\nu,0)$$
$$= \sum_{\kappa=0}^{\nu} \binom{\nu}{\kappa}(-1)^{\kappa}\left(\frac{1}{2\pi i}\int_{a+i}^{a+iT} \frac{\xi'}{\xi}(s)\zeta^{(\mu)}(s)\zeta^{(\kappa)}(s)\chi^{(\nu-\kappa)}(1-s)\,ds\right).$$

We may write this as

$$(55) \qquad I_1(\mu,\nu,0) = \sum_{\kappa=0}^{\nu} \binom{\nu}{\kappa}(-1)^{\kappa}\, I_{1\kappa}(\mu,\nu,0) ,$$

where

$$I_{1\kappa}(\mu,\nu,0) = \frac{1}{2\pi i}\int_{a+i}^{a+iT} \frac{\xi'}{\xi}(s)\zeta^{(\mu)}(s)\zeta^{(\kappa)}(s)\chi^{(\nu-\kappa)}(1-s)\,ds .$$

Now by Lemma 1.6

$$I_{1\kappa}(\mu,\nu,0) = \frac{(-1)^{\nu-\kappa}}{2\pi}\int_1^T \frac{\xi'}{\xi}(a+it)\zeta^{(\mu)}(a+it)\zeta^{(\kappa)}(a+it)$$
$$\cdot\ \chi(1-a-it)\left(\log\frac{t}{2\pi}\right)^{\nu-\kappa} dt$$
$$+\ O\left(\int_1^T \left|\frac{\xi'}{\xi}(a+it)\zeta^{(\mu)}(a+it)\zeta^{(\kappa)}(a+it)\right| t^{a-3/2}(\log t)^{\nu-\kappa-1}\,dt\right),$$

and if we use (46) to replace $\frac{\xi'}{\xi}(a+it)$ in the main term we obtain

$$I_{1\kappa}(\mu,\nu,0) = \frac{(-1)^{\nu-\kappa}}{2\pi} \int_1^T \frac{\zeta'}{\zeta}(a+it)\zeta^{(\mu)}(a+it)\zeta^{(\kappa)}(a+it)$$

$$\cdot\ \chi(1-a-it)(\log \frac{t}{2\pi})^{\nu-\kappa}\,dt$$

$$+ \frac{(-1)^{\nu-\kappa}}{2\pi} \int_1^T \frac{1}{2}\,\zeta^{(\mu)}(a+it)\zeta^{(\kappa)}(a+it)\chi(1-a-it)(\log \frac{t}{2\pi})^{\nu-\kappa+1}\,dt$$

$$+ \frac{(-1)^{\nu-\kappa}}{2\pi} \int_1^T \frac{\pi i}{4}\,\zeta^{(\mu)}(a+it)\zeta^{(\kappa)}(a+it)\chi(1-a-it)(\log \frac{t}{2\pi})^{\nu-\kappa}\,dt$$

$$+ O\left(\int_1^T \left|\zeta^{(\mu)}(a+it)\zeta^{(\kappa)}(a+it)\chi(1-a-it)\right|(\log t)^{\nu-\kappa}\frac{dt}{t}\right)$$

$$+ O\left(\int_1^T \left|\frac{\xi'}{\xi}(a+it)\zeta^{(\mu)}(a+it)\zeta^{(\kappa)}(a+it)\right| t^{a-3/2}(\log t)^{\nu-\kappa-1}\,dt\right) .$$

By (11), (20), and (21) the two error terms are easily seen to be $O(T^{a-1/2+\varepsilon})$. Thus we may write

$$(56) \qquad I_{1\kappa}(\mu,\nu,0) = I_{1\kappa 1} + I_{1\kappa 2} + I_{1\kappa 3} + O(T^{a-1/2+\varepsilon}) .$$

To treat $I_{1\kappa 1}$ we write $\frac{\zeta'}{\zeta}(s)\zeta^{(\mu)}(s)\zeta^{(\kappa)}(s) = \sum_{n=1}^{\infty} \frac{B_n}{n^s}$ (Re $s > 1$) so that

$$I_{1\kappa 1} = \frac{(-1)^{\nu-\kappa}}{2\pi} \int_1^T \left(\sum_{n=1}^{\infty} B_n n^{-a-it}\right)\chi(1-a-it)(\log \frac{t}{2\pi})^{\nu-\kappa}\,dt .$$

In order to use Lemma 1.5 to estimate this integral we must first show that $B_n \ll n^{\varepsilon}$ for any $\varepsilon > 0$. We have

$$|B_n| = \sum_{d|n} \Lambda(d) \sum_{e|(n/d)} (\log e)^{\mu} (\log \frac{n}{ed})^{\kappa}$$

$$\ll (\log n)^{\mu+\kappa} \sum_{d|n} \Lambda(d) \sum_{e|(n/d)} 1$$

$$= (\log n)^{\mu+\kappa} \sum_{d|n} \Lambda(d) d(\tfrac{n}{d})$$

$$\ll n^{\varepsilon/2} (\log n)^{\mu+\kappa} \sum_{d|n} \Lambda(d) = n^{\varepsilon/2} (\log n)^{\mu+\kappa+1}$$

$$\ll n^{\varepsilon} \quad .$$

Hence

$$(57) \qquad I_{1\kappa 1} = (-1)^{\nu-\kappa} \sum_{1\leq n\leq T/2\pi} B_n (\log n)^{\nu-\kappa} + O(T^{a-1/2} (\log T)^{\nu-\kappa}) \quad .$$

Now by Lemma 1.9

$$\sum_{1\leq n\leq T/2\pi} B_n = \frac{(-1)^{\mu+\kappa+1} \mu!\kappa!}{(\mu+\kappa+2)!} \frac{T}{2\pi} (\log \frac{T}{2\pi})^{\mu+\kappa+2} + O(T(\log T)^{\mu+\kappa+1}) .$$

Therefore we have from Lemma 1.11 that

$$\sum_{1\leq n\leq T/2\pi} B_n (\log n)^{\nu-\kappa} = \frac{(-1)^{\mu+\kappa+1} \mu!\kappa!}{(\mu+\kappa+2)!} \frac{T}{2\pi} (\log \frac{T}{2\pi})^{\mu+\nu+2} + O(T(\log T)^{\mu+\nu+1})$$

This and (57) lead to

(58) $$I_{1\kappa 1} = \frac{(-1)^{\mu+\nu+1}\mu!\kappa!}{(\mu+\kappa+2)!}\frac{T}{2\pi}(\log \frac{T}{2\pi})^{\mu+\nu+2}$$

$$+ O(T(\log T)^{\mu+\nu+1}) + O(T^{a-\frac{1}{2}}(\log T)^{\nu-\kappa+1}) .$$

We now estimate $I_{1\kappa 2}$. Write $\zeta^{(\mu)}(s)\zeta^{(\kappa)}(s) = \sum_{n=1}^{\infty} \frac{A_n}{n^s}$ (Re $s > 1$) . Then

$$I_{1\kappa 2} = \frac{(-1)^{\nu-\kappa}}{2\pi}\int_1^T \left(\frac{1}{2}\sum_{n=1}^{\infty} A_n n^{-a-it}\right)\chi(1-a-it)(\log \frac{t}{2\pi})^{\nu-\kappa+1}\, dt .$$

Since

$$|A_n| = \sum_{d|n} (\log d)^{\mu}(\log \frac{n}{d})^{\kappa} \ll (\log n)^{\mu+\kappa} \sum_{d|n} 1$$

$$= (\log n)^{\mu+\kappa} d(n) \ll (\log n)^{\mu+\kappa} n^{\varepsilon/2}$$

$$\ll n^{\varepsilon} ,$$

we have by Lemma 1.5

$$I_{1\kappa 2} = \frac{(-1)^{\nu-\kappa}}{2}\sum_{1\le n\le T/2\pi} A_n(\log n)^{\nu-\kappa+1} + O(T^{a-\frac{1}{2}}(\log T)^{\nu-\kappa+1}) .$$

From Lemma 1.8

(59) $$\sum_{1\le n\le T/2\pi} A_n = \frac{(-1)^{\mu+\kappa}\mu!\kappa!}{(\mu+\kappa+1)!}\frac{T}{2\pi}(\log \frac{T}{2\pi})^{\mu+\kappa+1}$$

$$+ O(T(\log T)^{\mu+\kappa}) .$$

It follows from this and Lemma 1.11 that

$$\sum_{1\le n\le T/2\pi} A_n(\log n)^{\nu-\kappa+1} = \frac{(-1)^{\mu+\kappa}\mu!\kappa!}{(\mu+\kappa+1)!}\frac{T}{2\pi}(\log \frac{T}{2\pi})^{\mu+\nu+2}$$

$$+ O(T(\log T)^{\mu+\nu+1}) \quad .$$

Thus

$$(60) \quad I_{1\kappa 2} = \frac{(-1)^{\mu+\nu}\mu!\kappa!}{2(\mu+\kappa+1)!}\frac{T}{2\pi}(\log \frac{T}{2\pi})^{\mu+\nu+2} + O(T(\log T)^{\mu+\nu+1})$$

$$+ O(T^{a-\frac{1}{2}}(\log T)^{\nu-\kappa+1}) \quad .$$

We treat $I_{1\kappa 3}$ in a manner analogous to $I_{1\kappa 2}$. Clearly

$$I_{1\kappa 3} = \frac{(-1)^{\nu-\kappa}}{2\pi}\int_1^T \left(\frac{\pi i}{4}\sum_{n=1} A_n n^{-a-it}\right)\chi(1-a-it)(\log \frac{t}{2\pi})^{\nu-\kappa}\, dt \; .$$

Hence by Lemma 1.5

$$I_{1\kappa 3} = \frac{(-1)^{\nu-\kappa}\pi i}{4}\sum_{1\le n\le T/2\pi} A_n(\log n)^{\nu-\kappa}$$

$$+ O(T^{a-\frac{1}{2}}(\log T)^{\nu-\kappa}) \quad .$$

Furthermore, from Lemma 1.11 and (59) we have

$$\sum_{1\le n\le T/2\pi} A_n(\log n)^{\nu-\kappa} = \frac{(-1)^{\mu+\kappa}\mu!\kappa!}{(\mu+\kappa+1)!}\frac{T}{2\pi}(\log \frac{T}{2\pi})^{\mu+\nu+1}$$

$$+ O(T(\log T)^{\mu+\nu}) \quad .$$

Thus

$$(61)\quad I_{1\kappa 3} = \frac{(-1)^{\mu+\nu}\mu!\kappa!}{(\mu+\kappa+1)!}\,\frac{\pi i}{4}\,\frac{T}{2\pi}(\log \frac{T}{2\pi})^{\mu+\nu+1}$$

$$+ O(T(\log T)^{\mu+\nu}) + O(T^{a-\frac{1}{2}}(\log T)^{\nu-\kappa}) .$$

We now combine (56), (58), (60), and (61) and take $a = \frac{5}{4}$ to obtain

$$I_{1\kappa}(\mu,\nu,0) = \frac{(-1)^{\mu+\nu}\mu!\kappa!}{(\mu+\kappa+1)!}\,(\frac{1}{2} - \frac{1}{\mu+\kappa+2})\frac{T}{2\pi}(\log\ T)^{\mu+\nu+2}$$

$$+ O(T(\log T)^{\mu+\nu+1}) .$$

Substituting this into (55), we find that

$$I_1(\mu,\nu,0) =$$

$$\left((-1)^{\mu+\nu}\mu!\nu! \sum_{\kappa=0} \frac{(-1)^{\kappa}}{(\mu+\kappa+1)!(\nu-\kappa)!}\,(\frac{1}{2} - \frac{1}{\mu+\kappa+2})\right)\frac{T}{2\pi}(\log\ T)^{\mu+\nu+2}$$

$$+ O(T(\log T)^{\mu+\nu+1}) .$$

Inserting this into (54) and taking $a = \frac{5}{4}$, we conclude that

$$\sum_{1\le\gamma\le T} \zeta^{(\mu)}(\rho)\zeta^{(\nu)}(1-\rho) = A(\mu,\nu)\frac{T}{2\pi}(\log\ T)^{\mu+\nu+2}$$

$$+ O(T(\log T)^{\mu+\nu+1}) ,$$

where

$$A(\mu,\nu) = (-1)^{\mu+\nu}\mu!\nu!\left(\sum_{\kappa=0}^{\nu} \frac{(-1)^{\kappa}}{(\mu+\kappa+1)!(\nu-\kappa)!}\left(\frac{1}{2} - \frac{1}{\mu+\kappa+2}\right) + \sum_{\kappa=0}^{\nu} \frac{(-1)^{\kappa}}{(\nu+\kappa+1)!(\mu-\kappa)!}\left(\frac{1}{2} - \frac{1}{\nu+\kappa+2}\right)\right) .$$

This proves (5) of Theorem 1.2 for T large, $T \varepsilon S$. However, the restriction $T \varepsilon S$ is easily removed as it was in the proof of Theorem 1.1 in Section I.5.

It remains to note that by the reflection principle, if $\mu = \nu$ and the Riemann hypothesis is true, the left-hand side of (5) is

$$\sum_{1\le\gamma\le T} |\zeta^{(\nu)}(\tfrac{1}{2} + i\gamma)|^2 .$$

Thus (6) is true and the proof of Theorem 1.2 is complete.